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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled

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ABSTRACT

The motion of a classical charged particle in the field of a magnetic monopole with a specially chosen potential is studied from a symplectic-geometric point of view. Throughout we emphasize the global structure of the space of solutions and its symmetries. In particular, a systematic geometric algorithm is presented that constructs the first integrals of the motion.

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INTRODUCTION

A magnetic monopole is a magnetic charge. That is to say, a magnetic field that is spherically symmetric and falls off as the inverse of the square of the distance from the charge. Despite have never been found in nature, magnetic monopoles form a central part of the current understanding of theoretical physics, running the gamut of gauge theories to elementary particles to cosmology. This popularity is due in large measure to the insight gained into current theories by sophisticated but solvable examples. For some flavour of current work the reader is referred to Balachandran et.al.(1980), Barut et.al. (1971), Carrigan (1965), D'Hoker and Vinet (1983-1984), Hitchin (1982), Jaffe and Taubes (1980), Miller (1976), Moriyasu (1981), Sanders (1966), T'Hooft (1974), and Wu and Yang (1975-1976).

In this essay we add a specially chosen potential to this magnetic field and study the motion of a classical charged particle in this field (see Boulware et.al.(1976), Schonfeld (1980), and Zwanziger (1968)). This provides a one-parameter family of inequivalent Hamiltonian systems that all "look somewhat like" the Kepler problem (for brevity we now refer to this system as the monopole problem).

In the prolegomenon we give some introductory remarks on the equations of motion and then, following Abraham and Marsden, introduce the notion of a Hamiltonian system to give a global construction of the monopole problem. The reader having little sympathy with bundle

constructions and momentum mappings, that is to say the current vogue in mechanics, can proceed directly to chapter two. In defense of this construction, which will seem incredibly abstract and obfuscating, it seems only right to point out that this is the only approach I know of that not only does the construction globally, but preserves the physical interpretation of variables. We provide enough detail to correct a statement of Marsden (1981) on the introduction of charge in this construction.

In chapter one we develop a geometric algorithm based on complete lifts that finds conserved quantities of the motion. In particular, we find the (corrected cf. Schonfeld) generators of a symplectic $SO(4)$ action on negative energy orbits. A by-product of this symplectic approach is the explanation of why the monopole problem is Kepler-like. In this chapter, sections two and three are essentially original material.

Chapter two exploits techniques due to Souriau (1974) to integrate the equations of motion and determine the topology of energy surfaces. The idea that these techniques could be extended to the monopole problem is due to Hans Künzle.

Finally, in the eschata we collect some useful definitions, fix notation, and explain some notions from mechanics and geometry.

1. THE MODEL

A magnetic monopole has a magnetic field $\vec{B}$, with

$$\vec{B} = \frac{g}{r^3}\vec{r} \qquad\qquad \vec{r} \in \mathbf{R} \ , \ r = \|\vec{r}\| \ .$$

Now consider the equation of motion of a classical charged particle of charge e in the field of a magnetic monopole (with specially chosen potential and setting $\lambda = eg$)

$$m\ddot{\vec{r}} = \frac{\lambda \vec{r} \times \dot{\vec{r}}}{r^3} - \nabla\left(-\frac{k}{r} + \frac{\lambda^2}{2mr^2} \right) . \qquad (1)$$

Note that when $\lambda = 0$ we have the Kepler problem. Now in (1) we define

$$\vec{J} = m\vec{r} \times \dot{\vec{r}} + \lambda\hat{r} \quad , \qquad\qquad \hat{r} = \frac{1}{r}\vec{r}$$

and calculate that $\dot{\vec{J}} = 0$.

Now

$$\hat{r} \cdot \vec{J} = \lambda$$

so the motion of the particle lies on a cone of pitch α with

$$\cos\alpha = \frac{\lambda}{J} \quad , \quad J = \|\vec{J}\|$$

Now define another vector $\vec{R}$ by

$$\vec{R} = \frac{1}{\sin\alpha}\left[\vec{r} - \hat{J}(\vec{r}\cdot\hat{J})\right] .$$

Then

$$\begin{aligned} R &= \|\vec{R}\| = r \\ \vec{R}\cdot\vec{J} &= 0 \\ m\vec{R}\times\dot{\vec{R}} &= \vec{J} \qquad \text{(1a)} \\ m\ddot{\vec{R}} &= -\nabla\left(\frac{-k}{R}\right) \end{aligned}$$

so we have transformed (1) into the Kepler problem!

At this stage one might be tempted to conjecture that the reason this works is that if we rewrote (1) in Hamiltonian form, then there is a canonical transformation between (1) and the Kepler problem. A local construction of a Lagrangian can be done if we find a potential $A = A_k dx^k$ for the electromagnetic field tensor $F = \frac{1}{2} F_{ij} dx^i \wedge dx^j$, where $F = dA$.

Now $F_{ij} = \varepsilon_{ijk} B^k$ and so

$$F = \frac{1}{2}\frac{g}{r^3}\varepsilon_{ijk}\, x^i dx^j \wedge dx^k \quad \text{in cartesian coordinates}$$

$$= g \sin \phi \, d\theta \wedge d\phi \qquad \text{in polar coordinates (see appendix).}$$

So, by inspection we may choose

$$A = g \cos \phi \, d\theta \; ,$$

and the Lagrangian has the form

$$L = \frac{1}{2} m \, v^2 + eA \cdot v + V(x)$$

and so the Hamiltonian is

$$\begin{aligned} H &= A \cdot v - L \\ &= \frac{1}{2m} \left(\hat{p} - eA\right)^2 + V(x) \end{aligned} \tag{2}$$

where $\hat{p}$ is the canonical momentum and the equations of motion are just Hamilton's equations. (see Crampin 1981, Havas 1957).

There is a serious objection to this procedure, however. The construction of the Lagrangian and Hamiltonian are only local, while the second order equation is globally well defined. In fact, one can prove that there is no global Lagrangian for this second-order equation. To remedy this we introduce the notion of a Hamiltonian system: (the basic geometrical ideas are well-covered in Abraham-Marsden, (1978) Souriau, (1970), Arnold (1978), Maclane, (1970)).

A Hamiltonian system is the triple (P,ω,h) where P is a manifold, ω is a symplectic form on P, and h is a function called the Hamiltonian. The equation of motion is given by $\iota_{X_h}\omega = dh$, where X_h is the symplectic gradient of h. From this geometric viewpoint we can formulate the monopole problem globally as

$$P = (\mathbf{R}^3 \setminus \{0\}) \times \mathbf{R}^3$$

$$\omega = \frac{1}{2}\frac{\lambda}{r^3}\varepsilon_{ijk}x^i dx^j \wedge dx^k + dx^k \wedge dp_k . \qquad (3)$$

$$h = \frac{p^2}{2m} + \frac{\lambda^2}{2mr^2} - \frac{k}{r} .$$

where $p_k = m\dot{x}^k$.

On the other hand, the Kepler problem is

$$\overline{P} = P$$

$$\Omega = dx^k \wedge dp_k \qquad (4)$$

$$H = \frac{p^2}{2m} - \frac{k}{r} .$$

The first result that may seem initially surprising in view of (1a) is that there is no canonical transformation between (3) and (4). In fact, a careful examination of the proof of this shows that the systems (3) are not equivalent by canonical transformation for any two distinct values of λ.

We thus have an interesting example of a one-parameter family of inequivalent Hamiltonian systems that all "look somewhat like" the Kepler problem. This is explained in I.2 using the geometrical reduction scheme of Marsden and Weinstein (This is the precise way of saying the classical notion of "freezing out the angular variables"). For more background and technical details one should consult Abraham-Marsden (1978), Marsden-Weinstein (1974), Giachetti (1981), Iwai (1980); and Marmo et.al. (1979).

It is legitimate to ask why the term $\frac{\lambda^2}{2mr^2}$ occurs in the potential in (1) . The reason is that this is precisely the condition we need to give us a conserved Lenz-like vector, just as in the Kepler problem. Furthermore, it again generates an action of SO(4) on our phase space by canonical transformations.

The rest of this section is devoted to explaining why the magnetic monopole problem with potential (hereafter the monopole) can be written globally as the Hamiltonian system (3) .

This is probably the most difficult section of the thesis, and, unless the reader wishes to see this, should probably proceed to the notes, keeping in mind that this construction is to show why the addition of a magnetic fields modifies the geometry of phase space (that is, the symplectic structure) but does not change the energy or our concept of momentum. The geometry is altered because now the velocities do not Poisson commute.

This construction is based on Sternberg (1977), Weinstein (1978) and Guillemin and Sternberg (1978). See also Marsden (1981). The relevant background on principal bundles and connections can be found in Choquet-Bruhat (1977), Kobayashi-Nomizu (1963), Poor (1981), Sternberg (1963), or Spivak (1974). For connections with monopoles see Cant (1981). Somewhat different approaches are in Duval and Horvathy (1982) and Montgomery (1983).

The current model for electromagnetism is a principal bundle M over space or space-time Q with structure group $G = U(1)=SO(2)$, with the electromagnetic field tensor F occuring as the curvature form of a connection on M. In this thesis we are interested in non-relativistic mechanics so we treat Q as space. That is to say, $Q = \mathbf{R}^3 \setminus \{0\}$. We also treat the connection on M as representing a vector potential for the magnetic field. We now want to build an associated symplectic manifold P, with a Hamiltonian, that is the Hamiltonian system for the monopole.

We choose local coordinates on the cotangent bundle of M, T^*M, as (q,α,z,ζ) (a fibre chart) where

$q \in Q$

$\alpha \in G$

$z \in \mathbf{R}^{3*}$

$\zeta \in g^*$, where g is the Lie algebra of G $(g \approx \mathbf{R}\ )$

The canonical forms on T^*M have the local expression

$$\theta_M = z_k dq^k + \zeta d\alpha$$

$$\omega_M = dq^k \wedge dz_k + d\alpha \wedge d\zeta \ .$$

The right action of G on M lifts to a symplectic right action on T^*M with the momentum map (see appendix)

$$J_M: T^*M \to g^*: (q,\alpha,z,\zeta) \to \zeta$$

To this right action we may associate the left action whose momentum map is $-J_M$. Now consider the Hamiltonian G space (see Abraham-Marsden (1978) p. 276) $S = T^*G$ with the left G action and canonical symplectic structure. The momentum map on S is $J_G: S \to \mathbf{R}: (\delta,e) \to (e)$ in a local fibre chart. Now the momentum map J on $T^*M\times S$ is just $J = -J_M + J_G$. We construct the reduced manifold $P = (T^*M\times S)_0$ as $P = J^{-1}(0)/G_0$ where G_0 is the isotropy group of 0: $G_0 = \{\alpha \in G |\ Ad^*_{\alpha^{-1}}(0) = 0\} = G$.

Thus we have an inclusion map i:

$$i: P \to T^*M\times S$$

which has the local form

$$(q,p) \to (q,0,p,-e,0,e)$$

Note that now $\{e\}$ is a co-adjoint orbit of G in g^*, which we will identify as the charge of the particle. The symplectic form $\omega_M+\omega_G$ on $T^*M\times S$ pulls back via i to give the induced form $i^*(\omega_M+\omega_G) = \omega_P$ on p which has the local form $dq^k \wedge dp_k$.

To get a map from P to T^*Q we pick a connection on M. In a local chart for TM we have $(q^k,\alpha,\partial_k,\partial_\alpha)$ while a local chart for TQ is (q^k,∂_k). A connection is a way of specifying a horizontal subspace of $T_{(q,\alpha)}M$ which we define by the map $\gamma: T_qQ \to T_{(q,\alpha)}M$

$$\partial_k \to \partial_k + A_k \partial_\alpha$$

whose dual $\gamma_q^*: T^*_{(q,\alpha)}M \to T^*_q Q$ takes

$$dq^k \to dq^k$$

$$d\alpha \to A_k dq^k.$$

Putting all the cotangent spaces together we get a map

$$\gamma^*: T^*M \to T^*Q: \quad (q,\alpha,z,\zeta) \to (q,y)$$

where $y_k = z_k + A_k\zeta$.

Given a Hamiltonian H on T^*Q we now pull it back to P to get a Hamiltonian $H_\gamma = h = i^*(H\,\gamma^*)$. On T^*Q we set $H = \frac{1}{2m}\,y^2 + V(q)$ and so h has the local form

$$h = \frac{1}{2m}(p-eA)^2 + V(q).$$

Making the transformation $p \to p + e\alpha$ we find that we can write the system globally as

$$h = \frac{1}{2m}\,p^2 + V(x)$$

$$\omega = \frac{1}{2}\,\frac{\lambda}{r^3}\,\varepsilon_{ijk}q^i dq^j \wedge dq^k + dq^k \wedge dp_k$$

where (q,p) are cartesian coordinates on $P \approx T^*Q$.

2. NOTES:

(A) It is also interesting to see why the potential V was chosen the way it was. The reason is primarily aesthetic. It turns out that this is the condition we need so that we have an $SO(4)$ symmetry of the Hamiltonian. This potential does not come from any field theory that I am aware of, although a potential differing by a factor of two in the $\frac{1}{r^2}$ term does come from the Newtonian theory of Kunzle (1972) as a non-relativistic limit of the Newton-Cartan equations in Taub-Nut space. Further discussion is in Schonfeld (1980).

(B) We now want to look at the cohomology class of ω and note some interesting consequences and implications. We set $\hat{\omega} = \omega + d\theta_0$, so we clearly have that $\hat{\omega}$ and ω are cohomologous. Furthermore, $\hat{\omega}$ can be thought of as a form on Q instead of $P = T^*Q$. Now S^2, the unit sphere in three-space, is a deformation retract of Q so we may evaluate the cohomology class of $\hat{\omega}$ on S^2. This induced form on S^2 is just the restriction of $\hat{\omega}$ since there are no terms involving dr in $\hat{\omega}$. Thus the cohomology class of ω, call it $[\omega]$, is

$$[\omega] = \lambda \int_{S^2} \sin\phi \, d\phi \, d\theta$$

$$= 4\pi\lambda.$$

This further implies that there is no global formulation of the monopole problem as a Lagrangian system. This is in some sense the ultimate justification for the seemingly abstract approach to formulating the equations of motion. It is well-known in geometric quantization that for a classical problem to be quantizable the cohomology class of the symplectic structure must be an integral multiple of Planck's constant. In other words

$$[\frac{1}{h}\,\omega] \in H^2(P,Z).$$

This only happens if the charge λ satisfies

$$\lambda = \frac{n\hbar}{2}\,,$$

which is exactly the quantization condition of Dirac (1931).

CHAPTER ONE

We now proceed to systematically exploit the symmetry of the monopole with a symplectic methodology. In this way we avoid the rather ad hoc and 'coincidental' approaches of Boulware, Schonfeld, and Zwanziger, because the symmetry is then seen as naturally reflecting the intrinsic geometry of the system.

1. THE OBVIOUS SYMMETRY GROUP.

An obvious symmetry is an infiniteminal symplectomorphism generated by a vector field on P that

(OS1) is the complete lift of a vector field on Q,

(OS2) is tangent to an energy surface.

In the tangent formulation this condition reads as

$$L_{\tilde{\chi}}\omega = 0, \tag{1}$$

where $\tilde{\chi}$ is the complete lift of χ. In other words, if

$$\chi = f^k \partial_k ,$$

then $\tilde{\chi} = f^k\partial_k + v^\ell \partial_\ell f^k \partial_k$. Now $L_{\tilde{\chi}}\omega = \iota_{\tilde{\chi}} d\omega + d\iota_{\tilde{\chi}}\omega = d\iota_{\tilde{\chi}}\omega$ since ω is closed.

Substituting, we get

$$L_{\hat{X}}\omega = \partial_{(j}f_{k)}dq^j\wedge dv^k + \partial_a(f^k\omega_{kb} - v^\ell\partial_\ell f_b)dq^a\wedge dq^b.$$

To satisfy the condition $L_{\hat{X}}\omega = 0$ we must have that the individual coefficients vanish. We now need the following:

Lemma: The condition $\partial_{(j}f_{k)} = 0$ implies f is an isometry.

Proof: Suppose we have $Df + Df^T = 0$ for all u and v, that is

$$\langle Df(u),v\rangle + \langle u,Df(v)\rangle = 0.$$

By differentiating both sides, we get

$$\langle D^2f(u,w),v\rangle + \langle u,D^2f(v,w)\rangle = 0 \quad \text{for all} \quad u,v,w.$$

By playing with these conditions, we derive that

$$\begin{aligned}\langle D^2f(u,w),v\rangle &= -\langle u,D^2f(v,w)\rangle \\ &= -\langle D^2f(w,v),u\rangle\end{aligned}$$

$$= \langle D^2 f(v,u), w\rangle$$

$$= -\langle D^2 f(u,w), v\rangle.$$

Thus $D^2 f(u,v) = 0$ for all u and v. Therefore, we conclude that

$$f = A \cdot q + B ,$$

where $A + A^T = 0$.

The other terms give no further information. Inspection of the condition $L_{\hat{X}} E = 0$ shows that B is zero. This means that f is an element of $so(3)$, the Lie algebra of $SO(3)$.

We conclude that the obvious symmetry group of the monopole is $SO(3)$, with its standard linear action.

2. THE MOMENTUM MAPPING

We now wish to construct the conserved quantities associated with the obvious symmetry group. The technical tool by which this is done is called the momentum mapping. The reader not acquainted with this apparatus should consult the appendix.

First, let $\xi \in g = so(3)$. Next, solve the equation $d\hat{J}(\xi) = \iota_{\xi_P}\omega$, where ξ_P is the infinitesimal generator of action corresponding to ξ.

To do this, we let ξ_1, ξ_2, ξ_3 be the usual basis for g. Then

$$(\xi_k)_P = -\varepsilon_k{}^{\ell}{}_m q^m \partial_\ell - \varepsilon_{k\ell}{}^m p_m \dot{\partial}^\ell ,$$

so that

$$\iota_{(\xi_k)_P}\omega = -\varepsilon_k{}^{\ell}{}_m q^m \omega_{\ell s} dq^s + d[\varepsilon_{km}{}^{\ell} q^m p_\ell].$$

We wish to express the first term as an exact differential. Some fiddling with indices yields

$$-\varepsilon_k{}^{\ell}{}_m q^m \omega_{\ell s} dq^s = \lambda d[q^k \|q\|^{-1}],$$

since $\omega_{\ell s} = \frac{1}{2}\frac{\lambda}{q^3}\varepsilon_{\ell st} q^t$, we may write.

$$\hat{J}(\xi_k) = \lambda q^k \|q\|^{-1} + \varepsilon_{km}{}^{\ell} q^m p_\ell, \tag{2}$$

and

$$\hat{J}(\xi)\cdot(q,p) = \langle\lambda\|q\|^{-1}q + q\times p, \xi_p\rangle.$$

Using the identification of $so(3)$ with $(R^3,\times)$ and R^{3*} with R^3 by $\langle\ ,\ \rangle$, we get

$$J(q,p)\cdot\xi = \langle q\times p + \lambda\|q\|^{-1}q, \hat{\xi}\rangle, \tag{3}$$

which is the desired form of the momentum map.

Since $SO(3)$ is semi-simple, the co-adjoint cocycle associated to J vanishes and so J is Ad*-equivariant. Thus $\hat{J}$ is a homomorphism from g to the Lie algebra of functions under the Poisson bracket.

The derivative of J is

$$DJ_{(q,p)}\circ(v,w) = \lambda\|q\|^{-1}[v - \|q\|^2\langle q,v\rangle\cdot q] + q\times w + v\times p\ ,$$

which tells us that all values of J are regular.
Furthermore, if $q\times p \neq 0$, then the isotropy group $G_\mu = SO(2)$ acts freely on $J^{-1}(\mu)$.

In this case, the conditions for reduction to work are satisfied and we may conclude that there is a two dimensional phase space P_μ with a unique symplectic form ω_μ. In other words,

$$P_\mu = J^{-1}(\mu)/G_\mu \quad \text{has a unique} \quad \omega_\mu \,.$$

To examine the reduced system we work in polar coordinates, where the symplectic form is

$$\omega = \lambda \sin\phi \; d\phi\wedge d\theta + dr\wedge dp_r + d\theta\wedge dp_\theta + d\phi\wedge dp_\phi \,, \tag{4}$$

and the momentum integrals are

$$J^1 = \lambda \cos\theta \sin\phi - \cos\theta \cot\phi \; p_\theta - \sin\theta \; p_\phi \,,$$

$$J^2 = \lambda \sin\theta \sin\phi - \sin\theta \cot\phi \; p_\theta + \cos\theta \; p_\phi \,, \tag{5}$$

$$J^3 = \lambda \cos\phi + p_\theta$$

We reduce at $\mu = (0,0,L)$ and note that the following relations hold:

$$p_\phi = \cos\theta \; J^2 - \sin\theta \; J^1 = 0$$

$$p_\theta = \lambda \sin\phi \tan\phi.$$

These imply that

$$p_\theta = L - \lambda \cos \phi$$

$$= L(1 - \lambda^2 L^{-2}).$$

After a little algebra we get

$$h_\mu = \frac{1}{2m}\left((p_r)^2 + \frac{L^2}{r^2}\right) - \frac{k}{r}\,, \tag{6}$$

$$\omega_\mu = dr \wedge dp_r\,,$$

as the Hamiltonian structure on the reduced phase space

$$P_\mu = T^*R^+.$$

We note that the reduced system is identical to the reduced Kepler system.

3. THE HIDDEN SYMMETRY

A hidden symmetry is an infinitetimal symplectomorphism on P that

(HS1) is not an obvious symmetry,

(HS2) is tangent to an energy surface.

Since the set of all such vector fields forms an infinite dimensional Lie algebra, we will restrict ourselves to a geometrically meaningful finite dimensional subset which, in keeping with the approach for obvious symmetries, is again generated by complete lifts.

To construct such vector fields take the complete lift to P of a type $(1,1)$ tensor on Q and contract it with the second order equation. Add this vector field to the complete lift of a type $(1,0)$ tensor (a vector field) on Q to get a candidate for a generator of a hidden symmetry.

That is, let $f \in \mathscr{T}^1_1(Q)$ so that f has the local form $f = f^k_\ell \partial_k \otimes dq^\ell$. Then the complete lift of f, denoted f^C, has the property that $f^C \in \mathscr{T}^1_1(TQ)$, and has the local form

$$f^C = f^k_\ell \partial_k \otimes dq^\ell + f^k_\ell \partial_{\dot{k}} \otimes dv^\ell + v^k \partial_k f^\ell_m \partial_\ell \otimes dq^m \; . \tag{7}$$

The contraction then has the local form

$$\iota_{\chi_L} f^C = f^k_\ell v^\ell \partial_k + (v^j v^k \partial_j f^\ell_k + \xi^k f^\ell_k)\partial_{\dot{\ell}} \ , \tag{8}$$

where

$$\chi_L = v^k \partial_k + \xi^k \partial_{\dot{k}}$$

is the second order equation. We let T be a candidate to generate a hidden symmetry. That is,

$$T = \iota_{\chi_L} f^C + \tilde{\chi} \ , \tag{9}$$

where $\tilde{\chi}$ is the complete lift of a vector field χ on Q. We now wish to solve the equations

$$\begin{aligned} L_T \omega &= 0, \\ L_T E &= 0, \end{aligned} \tag{10}$$

where E is the energy.

With the usual methods of index gymnastics a straightforward, but long, calculation shows that the first equation implies that

$$f_{k\ell} = d_{k\ell m} q^m + e_{k\ell} \ ,$$

where $$f_{k\ell} = \delta_{km} f^m_\ell ,$$

$$d_{k\ell m} = d_{\ell k m} , \tag{11}$$

$$e_{k\ell} = e_{\ell k},$$

and $$d_{k\ell m} + d_{\ell m k} + d_{m k \ell} = 0.$$

Letting $\chi = g^k \partial_k$ we find also that

$$\partial_{(k} g_{j)} + \partial_{\dot{(k}} \xi^\ell f_{j)\ell} = 0,$$

$$\partial_k [v^m \partial_m g_\ell + \partial_{\dot{\ell}} ((f_{mn} v^n + g_m) \xi^m)] = 0, \tag{12}$$

where again we have lowered indices with δ.

For the Kepler problem we know that we can solve these equations (11) and (12) with 3 different solutions

$$d_{jk\ell m} = (\delta_{jm} \delta_{k\ell} - \delta_{j(k} \delta_{\ell)m}),$$

$$e_{jk\ell} = 0,$$

$$g_{jk} = 0,$$

where j acts like a tensor index. Using these as our starting point we find (after much manipulation) that we can again solve these for the monopole problem with the values

$$d_{jk\ell m} = (\delta_{jm}\delta_{k\ell} - \delta_{j(k}\delta_{\ell)m}),$$

$$e_{jk\ell} = 0, \tag{13}$$

$$g_{jk} = \frac{\lambda}{r}\ \varepsilon_{jk\ell}q^{\ell}.$$

If we let χ_{R^j} be the vector field formed with these values of $d_{jk\ell m}$, $e_{jk\ell}$ and g_{jk}, and find R^j which is defined by the relation

$$dR^j = \iota_{\chi_{R^j}}\omega ,$$

we calculate that

$$R^j = m\varepsilon^j{}_{km}v^k J^m - \frac{mk}{r}q^k, \tag{14}$$

and on the cotangent bundle that

$$R^j = \varepsilon^{jk}{}_m p_k J^m - \frac{mk}{r}q^k . \tag{15}$$

From these it is straightforward to show that the Poisson brackets

satisfy

$$\{J^k, R^\ell\} = \varepsilon^{k\ell}{}_m R^m ,$$
$$\{R^k, R^\ell\} = (-2mh)\varepsilon^{k\ell}{}_m J^m . \tag{16}$$

Before discussing these equations we first introduce some notation for constant energy surfaces:

$$\textstyle\sum_e := \{x \in T^*Q \,|\, H(x) = e\}.$$

We also define the open submanifolds of P:

$$\textstyle\sum_\pm := \bigcup_{e \gtrless 0} \sum_e .$$

On $\sum_\varepsilon$ we define

$$J^k_\varepsilon := J^k \,|\textstyle\sum_\varepsilon ,$$

$$K^k_\varepsilon := (\varepsilon 2mH)^{-1/2} R^k |\textstyle\sum_\varepsilon ,$$

$$\varepsilon = \pm .$$

Now on $\sum_\varepsilon$ we have the Poisson brackets:

$$\{J^k_\varepsilon J^\ell_\varepsilon\} = \varepsilon^{k\ell}{}_m J^m_\varepsilon \ ,$$

$$\{J^k_\varepsilon, K^\ell_\varepsilon\} = \varepsilon^{k\ell}{}_m K^m_\varepsilon \ , \tag{17}$$

$$\{K^k_\varepsilon, K^\ell_\varepsilon\} = -\varepsilon\varepsilon^{k\ell}{}_m J^m_\varepsilon \ .$$

So we see that under the Poisson bracket the functions L^k_ε and K^ℓ_ε generate the Lie algebras so(3,1) and so(4) on $\sum_e$ for $\varepsilon = +$ and $\varepsilon = -$ respectively.

For the sequel we are only interested in the case $\sum_-$, so we abuse notation and remember that from now on J^k means J^k_ε and K^j means K^j_ε.

It is not difficult to show that the symplectic gradients of the J^ks and K^js span an energy surface. This means that for any $e < 0$, SO(4) has a symplectic action on P , which is transitive on $\sum_e$. In the next chapter we will try to realize this action.

4. Notes:

(A) The obvious symmetry of the monopole was first discovered by Poincaré (1896) while explaining some seemingly paradoxical observations relating to experiments done on a Crooke's tube. In this paper Poincaré proved that, in the absence of a potential, an electrically charged particle has its motion confined to the surface of a cone, where the monopole is situated at the vertex. Furthermore, he showed that the particle motion is a geodesic, where the metric is the Euclidean metric induced by the imbedding of the cone in $\mathbf{R}^3$.

(B) This sort of result was investigated by Boulware et al (1976), who opened these cones out into a plane to discover that the motion on the cone corresponded to motion in an inverse square law potential in the corresponding plane. We call such a map an umbrella map, since the opening of the cone is similar to the way an umbrella is opened (we shall have more to say on this map later).

(C) The hidden symmetry was first discovered by Zwanziger (1968) who introduced the extra term in the potential so that the Hamiltonian would look simpler in polar coordinates.

(D) Schonfeld (1980) combined the extra potential term of Zwanziger and the umbrella map of Boulware to display the hidden symmetry. This works because the monopole problem is transformed into the Kepler problem.

(E) The preceeding results, while very pretty, must be viewed as incomplete since none of the preceeding authors discuss the symplectic structure. Further, the umbrella map must be viewed as suspect from the viewpoint of symplectic geometry, since the umbrella map is not symplectic. In other words, when confronted with a non-canonical transformation we say our Hamiltonian creed: "What is canonical is important and what is important is canonical".

(F) The cohomology class of ω shows us that there can be no symplectic map between the Kepler problem and the monopole problem. However, the reduction process shows us that it is possible for two inequivalent Hamiltonian systems to be acted on by the same group, with the same action, reduced at regular values and have the resulting reduced systems equivalent. Abraham and Marsden (1978) says that it is also possible to reduce exact symplectic manifolds and get non-exact symplectic forms on the reduced spaces. Thus it would seem that there are no cohomology theorems for momentum mappings.

(G) Considering both (E) and (F), we would conclude that the geometric understanding of the umbrella map is on the level of the reduced systems, and not on the phase space level.

(H) The construction of the hidden symmetry generators can be extended to $f \in \mathcal{T}^1_n(Q)$ for arbitrary positive n as follows:

Let $T = \iota_{X_L} \iota_{X_L} \cdots \iota_{X_L} f^C$ where f^C is again the complete lift of f. That is, $f^C \in \mathcal{T}^1_n(TQ)$. Now T has the local coordinate form for f symmetric in the covariant part:

$$T = v^{k_1} \cdots v^{k_n} f^m_{k_1 \cdots k_n} \partial_m + \left(v^{k_1} \cdots v^{k_n} v^\ell \partial_\ell f^m_{k_1 \cdots k_n} + \right.$$

$$\left. n v^{k_1} \cdots v^{k_{n-1}} \xi^{k_n} f^m_{k_1 \cdots k_n} \right) \partial_{\dot m} .$$

To see that such constructions are needed in mechanics, consider the n-dimensional Toda lattice, where a hidden symmetry exists as a polynomial of degree n for every n.

(I) The outstanding problem remains, however. That is, to give an algorithmic understanding of when one can extend such procedures to generate finite dimensional Lie groups of

symmetries that include the obvious symmetry group. The situation is not all black however, for we see that if such a group is generated by vector fields that are the symplectic gradients of functions polynomial in the v^ks, then such a construction is possible. Such an approach is also promising if one wishes to construct dynamical groups (see Marle (1976)).

(J) The reason for working on the tangent bundle instead of the cotangent bundle is essentially simplicity. As an example, compare the discussions of polynomial observables in Woodhouse (1980) and Kostant (1974). The essentials are not the complete story, for one has the freedom to construct lifts using either the canonical or the charged symplectic structure, and it is not a priori clear how this can be sorted out.

(K) The approach I have taken towards symmetries is due to the general philosophy that when one is working with geometric structures, one should only use constructions that depend on the given geometry. Concretely, the geometry given in the system has two facets: the symplectic structure of phase space and the metric structure of the individual cotangent spaces. I do not invoke the accidental Euclidean structure of phase space. I have endeavoured to provide a reasonable comprehensive account of the recent work in symmetries in

mechanics in general and applications to the Kepler problem in particular. The reader is invited to persue the bibliography and note the different approaches taken. Lie transforms, Jacobi fields or Lagrangian dynamics (and others!) are represented. For instance, one can see Andrie and Simms (1972), Kunzle (1969), Levy-Leblond (1970), Mariwalls (1975), Rogers (1973) and Wolf (1977).

CHAPTER TWO

1. REALIZATION

This seems a suitable point to review our journey so far, and preview what is yet to come. In the προλεγομενον we constructed the phase space, symplectic structure, and the Hamiltonian so that we knew precisely what mechanical system we were studying. Then, we set out to systematically find the symmetries of the system, both obvious and hidden. Now, we try to realize the symmetry. By this we mean that we try to find coordinates so that the symmetry now appears as more or less self-evident. Along the way, we also see how the symmetry gives a very elegant way to integrate the equations of motion.

Due to the algebraic and computational methods used in this section, we first transform to dimensionless variables to avoid excessive bulk in calculation and formulae.

For $k, \lambda > 0$ put

$$\vec{r} = \frac{km}{\lambda^2}\vec{q} \qquad r = \|\vec{r}\|$$

$$\vec{v} = \frac{\lambda}{mk}\vec{p} \qquad v = \|\vec{v}\|$$

$$\tau = \frac{mk^2}{\lambda^3}t \qquad \dot{} = \frac{d}{dt} \; .$$

We then have the relations

$$\vec{v} = \frac{d\vec{r}}{d\tau}$$

$$h = \frac{mk^2}{\lambda^2} H = \frac{1}{2} v^2 - \frac{1}{r} + \frac{1}{2r^2}$$

$$\vec{j} = \lambda\vec{J} = \vec{r}\times\vec{v} + \frac{1}{r}\vec{r}$$

$$\vec{k} = |k|\vec{R} = \vec{v}\times\vec{j} - \frac{1}{r}\vec{r}$$

$$\omega = \frac{1}{j}\omega_\lambda = \frac{1}{2} r^{-3}\varepsilon_{k\ell m} r^k dr^\ell\, dr^m + dr_k\, dv^k .$$

A short calculation gives

$$\vec{j}\cdot\vec{j} = r^2 v^2 - (\vec{r}\cdot\vec{v})^2 + 1$$

$$\vec{r}\cdot\vec{j} = r$$

$$\vec{r}\cdot\vec{k} = j^2 - r - 1$$

$$\vec{v}\cdot\vec{j} = \frac{1}{r}(\vec{r}\cdot\vec{v}) \tag{A}$$

$$\vec{v}\cdot\vec{k} = -\frac{1}{r}(\vec{r}\cdot\vec{v})$$

$$\vec{j}\cdot\vec{k} = -1$$

$$\vec{k}\cdot\vec{k} = 2h(j^2-1) + 1 .$$

The dimensionless form of the equations of motion is

$$\dot{\vec{r}} = \vec{v}$$

$$\dot{\vec{v}} = -\frac{1}{r^3}(\vec{r}-\vec{j}).$$

Imitating Souriau (1974), we now define a new parameter s (the Levi-Civita time) by

$$s = \vec{r}\cdot\vec{v} - 2h\tau$$

so that

$$\frac{ds}{dt} = \frac{1}{r}\ ;\ \text{in other words}\ \frac{d}{ds} = r\,\frac{d}{dt}\ .$$

With the use of the previous formulae, we derive

$$\tau' = r \tag{1}$$

$$\tau'' = \vec{r}\cdot\vec{v} \tag{2}$$

$$\vec{r}\,' = r\vec{v} \tag{3}$$

$$\vec{r}\,'' = (\vec{r}\cdot\vec{v})\vec{v} - \frac{1}{r}\,\vec{r} + \frac{1}{r}\,\vec{j} \tag{4}$$

$$= (\vec{r}\cdot\vec{v})\vec{v} + \left(\frac{1}{r^2} - \frac{1}{r}\right)\vec{r} + \frac{1}{r}\,\vec{r}\times\vec{v} \tag{5}$$

where ' denotes d/ds.

Using (1), (2), and (4) we see

$$\vec{r}\,'' = \frac{1}{\tau'}\,[\tau''\vec{r}\,' - \vec{r} + \vec{j}], \tag{6}$$

while differentiating (2) gives

$$\tau''' = 1 + 2h\tau'\ . \tag{7}$$

Differentiating (6) and using (7) yields

$$\vec{r}''' = 2h\vec{r}' \; . \tag{8}$$

Now we call $\xi = \begin{pmatrix} \tau \\ \vec{r} \end{pmatrix}$, $\Xi^T = (\xi, \xi', \xi'', \xi''')$. The motivation is because now

$$\begin{aligned} \tau'''' &= 2h\tau'' \\ \vec{r}'''' &= 2h\vec{r}'' \end{aligned} \tag{9}$$

which, when translated into (ξ, Ξ) form, becomes the linear matrix differential system

$$\Xi' = A\Xi, \tag{10}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2h & 0 \end{pmatrix} .$$

The fundamental solution to the initial value problem (10) with $\Xi(s_0) = I_4$ is

$$\Xi(s) = e^{(s-s_0)A} \, ,$$

so we have reduced the integration of the equations of motion to exponentiating the matrix A.

Using the formulas (1) through (10) we can easily calculate the relations

$$\vec{r}''^2 - 2h\tau''^2 = 1 + 2h$$

$$\vec{r}'''^2 - 2h\tau'''^2 = -2h(1+2h)$$

$$\vec{r}''\cdot\vec{r}''' - 2h\tau''\tau''' = 0$$

For $-\frac{1}{2} \leq h < 0$ we define ψ by $\psi^2 = -2h$, where ψ is positive.

Now define X and Y by

$$X = \begin{pmatrix} \psi\tau'' \\ \vec{r}'' \end{pmatrix} \qquad Y = \begin{pmatrix} \tau'' \\ \frac{1}{\psi}\,\vec{r}'' \end{pmatrix} .$$

We see that X and Y satisfy

$$\|X\| = 1 - \psi^2$$

$$\|Y\| = 1 - \psi^2$$

$$X \cdot Y = 0 ,$$

so for $0 < \psi < 1$ an energy surface $\sum_\psi$ has the topology of $S^3 \times S^2$. For $\psi = 1$ it is easy to see $\sum_{\psi=1} = S^2$. Summing this discussion up formally, we get the

THEOREM: (TOPOLOGY OF AN ENERGY SURFACE)

$$\sum_\psi = \begin{cases} S^3 \times S^2 & 0 < \psi < 1 \\ S^2 & \psi = 1 \end{cases} .$$

If we write

$$X = \begin{pmatrix} X^0 \\ \vec{X} \end{pmatrix} \qquad Y = \begin{pmatrix} Y^0 \\ \vec{Y} \end{pmatrix}$$

we can invert the mapping

$$(\vec{r},\vec{v}) \to \left((\vec{X},\vec{Y}),\ (X^0\ ,Y^0,\psi)\right)$$

to get

$$\vec{r} = \frac{1}{\psi^2\mu^2}\left[(\mu^2 - Y^0)\vec{X} + X^0\vec{Y} - \psi\vec{X}\times\vec{Y}\right]$$

$$\vec{v} = \frac{-\psi}{1-Y^0}\,\vec{Y} .$$

We also calculate

$$\vec{j} = \frac{1}{\mu^2}\left[Y^0\vec{X} - X^0\vec{Y} + \frac{1}{\psi}\vec{X}\times\vec{Y}\right]$$

$$\vec{k} = \frac{-1}{\mu^2}\left[Y^0\vec{X} - X^0\vec{Y} + \psi\vec{X}\times\vec{Y}\right] ,$$

where $\mu^2 = 1 - \psi^2$.

With these variables we can easily describe $\sum_- = \bigcup_{0<\psi\leq 1} \sum_\psi$. If we put

$$A = \begin{pmatrix} X \\ \psi \\ 0 \end{pmatrix} \qquad B = \begin{pmatrix} Y \\ 0 \\ \frac{\psi}{j}\vec{j} \end{pmatrix}$$

then we have the relations

$$\|A\| = 1$$

$$\|B\| = 1$$

$$A\bullet B = 0.$$

Since $\vec{j}$ can be described in terms of X and Y, we then may conclude the

THEOREM: (TOPOLOGY OF $\sum_-$)

$$\sum_- = S^4_+\times S^2$$

where S^n_+ is the open upper hemisphere of S^n.

These theorems propose the two natural questions:

a) What is the symplectic structure on $\sum_-$?

b) How does $SO(4)$ act on $\sum_-$?

to which we now turn.

When examining the symplectic form, one rapidly discovers that the variables X and Y are quite simply the wrong coordinates to use. An alternative is to examine the Poisson structure on some related manifold.

We define $\vec{E}$ and $\vec{F}$ by

$$\vec{E} = \frac{1}{2}(\vec{j} + \frac{1}{\psi}\vec{k})$$
$$\vec{F} = \frac{1}{2}(\vec{j} - \frac{1}{\psi}\vec{k}).$$

Then these satisfy the relations

$$\begin{aligned} \{E^k, E^\ell\} &= \varepsilon^{k\ell}{}_m E^m \\ \{E^k, F^\ell\} &= 0 \\ \{F^k, F^\ell\} &= \varepsilon^{k\ell}{}_m F^m. \end{aligned} \tag{11}$$

For notational convenience we also set

$$E = \|\vec{E}\| \quad \text{and} \quad F = \|\vec{F}\|.$$

Some calculation then shows

$$\vec{r} = \frac{1}{A}\,[a\vec{E} + b\vec{F} + c\vec{E}\times\vec{F}] \tag{12}$$
$$\vec{v} = \frac{1}{B}\,[-\chi F\vec{E} + E\chi\vec{F} - \xi\vec{E}\times\vec{F}]$$

where

$$A = (\psi^2(1+\psi)F(EF-G))^{-1}$$

$$a = (F^2(\xi(1+\psi-\chi)+\chi^2))$$

$$b = [(EF-G)(\chi^2+\psi^2+\psi) - \chi^2 G - EF\xi(1+\psi-\chi)]$$

$$c = \chi F(\xi-(1+\psi-\chi))$$

$$B = (1-\xi)(EF-G)/\psi$$

$$G = \vec{E}\cdot\vec{F}$$

$$\chi = X^0$$

$$\xi = Y^0$$

$$\psi = 1/(E+F).$$

We also have the constraints

$$F = 1 + E$$

$$\chi^2 + \xi^2 = 2(EF-G)/(E+F)^2.$$

We introduce a new variable σ by

$$\chi = \psi\sqrt{2(EF-G)}\cos\sigma$$

$$\xi = -\psi\sqrt{2(EF-G)}\sin\sigma$$

so we work in a seven dimensional space and derive that

$$\begin{aligned}\{E^k,\sigma\} = \cos^2\sigma[(\tfrac{a}{A}\,d - \tfrac{\chi F}{B}\,e + \psi f)E^k \\ + (\tfrac{b}{A}\,d + \tfrac{\chi E}{B}\,e - \psi f)F^k \\ + (\tfrac{c}{A}\,d - \tfrac{\xi}{B}\,e)(\vec{E}\times\vec{F})^k],\end{aligned}$$

$$\{F^k,\sigma\} = -\{E^k,\sigma\},$$

where

$$d = \left(\frac{1}{2y} + \frac{\xi}{2\chi^2}\left((\frac{1-\xi}{y}) - (\frac{2y-1}{y^2})\right)\right)$$

$$e = \left(\frac{\xi}{2\chi\psi} - \frac{y\psi}{2\chi}\right)$$

$$f = \left(\frac{1}{2(1-\xi)} - \frac{\xi}{2\chi^2}\,(1+\frac{1}{1-\xi})\right)$$

$$y = (E+F)(E+F-\sqrt{2(EF-G)}\sin\sigma).$$

From all this we see that SO(4) acts as two independent copies of SO(3) on $\vec{E}$ and $\vec{F}$ in the standard way, but in a highly convoluted way in σ. However, at last it is possible to 'see why' SO(4) is in the magnetic problem, although the global topology can no longer be seen clearly.

Some concluding remarks are now in order. The presence of the magnetic field makes all calculations involving the symplectic form (or

dually, the Poisson structure) absolutely non-trivial. This is because the Poisson structure is

$$\{r^k, r^\ell\} = 0$$

$$\{r^k, v^\ell\} = \delta^{k\ell}$$

$$\{v^k, v^\ell\} = -r^{-3}\varepsilon^{k\ell}{}_s r^s .$$

One would hope that it would be possible to find a variable ν such that the Poisson brackets $\{E^k, \nu\}$, $\{F^k, \nu\}$ had a very pleasing appearance. However, there is no *a priori* reason why such a variable should exist.

2. Notes:

(A) The parameter s was first introduced by Levi-Civita (1906) to compactify the two-dimensional Kepler problem. The most systematic exploitation of s occurs in the work of Souriau. It is still completely amazing to me that it works so well in the monopole problem.

(B) In the regularized Kepler problem all negative energy manifolds have the topology of $S^3 \times S^2$. The appearance of rest points is completely unique to the monopole.

(C) For the variables $(\vec{E},\vec{F},\nu)$ one can find 'nice' Poisson structures. However, the presence of $\vec{E}\cdot\vec{F}$ terms prevents us from actually realizing any of these structures concretely as the monopole problem.

(D) One of the disappointing features of the monopole is the lack of a global symplectic potential. The 'disappointing' feature of this is that it prevents us from defining action-angle variables in any open set of configuration space. Weinstein has shown how to construct such variables for non-abelian groups (e.g. SO(3)), but it seems an open problem of how to generalize the construction to non-exact forms. Two resolutions seem possible. The first is to work on a U(1) bundle over phase space where a global action exists. This seems somewhat unappetizing. The second is to

somehow understand the momentum map more fully. For example, if one pulls back the Kirillov-Kostant-Souriau symplectic form on the co-adjoint orbit of SO(3) via the momentum map for the Kepler problem and subtracts it from the canonical symplectic form, one finds that the resulting presymplectic form is everywhere orthogonal to the angular momentum.

A.1 Notation and Defintions.

In this appendix we fix the notational conventions used in this thesis.

1. Polar Coordinates: The transformation from rectangular to polar coordinates is defined on an open subset of $\mathbf{R}^3$ by the map $T : (r,\theta,\phi) \to (q^1,q^2,q^3)$ given by

$$q^1 = r \cos \theta \sin \phi$$
$$q^2 = r \sin \theta \sin \phi$$
$$q^3 = r \cos \phi$$

Orthonormal (with respect to the standard inner product) bases for the tangent and cotangent spaces are given by

$$\left(\frac{\partial}{\partial q^1} , \frac{\partial}{\partial q^2} , \frac{\partial}{\partial q^3} \right)$$

$$\left(dq^1, \ dq^2, \ dq^3\right) \ .$$

Coordinate bases in the polar system are given by

$$\left(\frac{\partial}{\partial r} , \frac{\partial}{\partial \theta} , \frac{\partial}{\partial \phi} \right)$$

$$(dr, \ d\theta, \ d\phi) \ .$$

However, these are not orthonormal. Orthonormal, non-coordinate frames are given by

$$\left(\partial_{\hat{r}}\,,\ \partial_{\hat{\theta}}\,,\ \partial_{\hat{\phi}}\right)$$

$$\left(\theta^{\hat{r}},\ \theta^{\hat{\theta}},\ \theta^{\hat{\phi}}\right) .$$

These are related to the coordinate frame by the relations

$$\partial_{\hat{r}} = \frac{\partial}{\partial r}$$

$$\partial_{\hat{\theta}} = \frac{1}{r \sin \phi} \frac{\partial}{\partial \theta}$$

$$\partial_{\hat{\phi}} = \frac{1}{r} \frac{\partial}{\partial \phi}$$

$$\theta^{\hat{r}} = dr$$

$$\theta^{\hat{\theta}} = r \sin \phi \ d\theta$$

$$\theta^{\hat{\phi}} = r \ d\phi \quad .$$

In the body of the thesis we make use of the relations

$$dq^1 \wedge dq^2 \wedge dq^3 = r^2 \sin \phi \ dr \wedge d\phi \wedge d\theta$$

$$\frac{1}{2} \varepsilon_{k\ell m} q^m \ dq^k \wedge dq^\ell = r^3 \sin \phi \ d\phi \wedge d\theta \ .$$

2. Invariant Scalars: The usual action of $SO(3)$ on R^3 induces an action on R^3 which leaves the functions

$$r = \left(q^k q_k\right)^{1/2}$$
$$v = \left(v^k v_k\right)^{1/2}$$
$$\alpha = q^k v_k$$

invariant. This notation is needed in the calculations on local equivalence.

3. Summation Convention: The summation convention of Einstein is used consistently unless followed by the sign for no sum: $\not\Sigma$.

4. Differentiation Conventions: Our notation for differentiation is defined by the relations

$$\partial_k f = \frac{\partial f}{\partial q^k}$$

$$\partial_{\dot{k}} f = \frac{\partial f}{\partial v^k}$$

$$\partial^{\dot{k}} f = \frac{\partial f}{\partial p_k} \quad .$$

Similar notation is used for vector fields:

$$X^k \partial_k + Y^\ell \partial_{\dot{\ell}} = X^k \frac{\partial}{\partial q^k} + Y^\ell \frac{\partial}{\partial v^\ell} \quad .$$

5. <u>Background References:</u> For background material on differential geometry see the books by Abraham-Marsden-Ratiu, Kobayashi-Nomizu, Poor, and Spivak. Material on characteristic classes can be found in Bott-Tu, Milnor-Stasheff, and Vaisman. For mechanics one should consult Abraham-Marsden, Arnold and Souriau. However, the best single reference in the spirit of this thesis is the book by Woodhouse on geometric quantization.

6. Notions from Differential Geometry

We consistently follow the notation of Abraham-Marsden (1978). For a manifold Q we let TQ denote its tangent bundle and let T^*Q denote its cotangent bundle. $\mathcal{T}^m_n(Q)$ denotes the tensors of type (m,n) on Q. For any tensor $T \in \mathcal{T}^m_n(Q)$ we may define a tensor $T^C \in \mathcal{T}^m_n(TQ)$ called the complete lift of T. We give local coordinate descriptions (which are easily checked to be invariant) for types $(1,0)$ and $(1,1)$ in the body of the thesis. For the definitive treatise on this branch of differential geometry see Yano and Ishihara (1973).

If G is a Lie group acting on a manifold Q and the Lie algebra of G is g, then the infinitesimal generator of action on Q (a vector field) corresponding to $\xi \in g$ is denoted by ξ_Q. Good references for Lie groups are Abraham-Marsden (1978), Cohn (1957), Chevalley (1946), and Helgason (1978).

7. The Momentum Mapping.

Let (P,ω) be a connected symplectic manifold and $\Phi : G\times P \to P$ a symplectic action of the Lie group G on P. By this we mean that for each $a \in G$, $\Phi_a^* \omega = \omega$. Let G have Lie algebra g and dual g^*. We then call the map $J : P \to g^*$ a momentum mapping for the action Φ if for each $\xi \in g$

$$d\hat{J}(\xi) = \iota_{\xi_P} \omega$$

where $\hat{J}(\xi) : P \to \mathbf{R}$ is defined by $\hat{J}(\xi)(x) = J(x)\cdot\xi$. The important thing to remember is that this is simply the generalization of the geometric ideas behind translational and rotational invariance giving us linear and angular momentum. For more details see Abraham-Marsden (1978), Souriau (1970), Woodhouse (1980), Cushman (1974) and Atiyah-Bott (1984).

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